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Electrohydrodynamic wave-packet collapse and soliton instability for dielectric fluids in (2+1)-dimensions

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Abstract. A weakly nonlinear theory of wave propagation in two superposed dielectric fluids in the presence of a horizontal electric field is investigated using the multiple scales method in (2+1)-dimensions. The equation governing the evolution of the amplitude of the progressive waves is obtained in the form of a two-dimensional nonlinear Schrödinger equation. We convert this equation for the evolution of wave packets in (2+1)-dimensions, using the function transformation method, into an exponentional and a Sinh-Gordon equation, and obtain classes of soliton solutions for both the elliptic and hyperbolic cases. The phenomenon of nonlinear focusing or collapse is also studied. We show that the collapse is direction-dependent, and is more pronounced at critical wavenumbers, and dielectric constant ratio as well as the density ratio. The applied electric field was found to enhance the collapsing for critical values of these parameters. The modulational instability for the corresponding one-dimensional nonlinear Schrödinger equation is discussed for both the travelling and standing waves cases. It is shown, for travelling waves, that the governing evolution equation admits solitary wave solutions with variable wave amplitude and speed. For the standing wave, it is found that the evolution equation for the temporal and spatial modulation of the amplitude and phase of wave propagation can be used to show that the monochromatic waves are stable, and to determine the amplitude dependence of the cutoff frequencies.

PACS. 47.20.-k Hydrodynamic stability – 52.35.Sb Solitons; BGK modes – 42.65.Jx Beam trapping, self-focusing and defocusing; self-phase modulation – 47.65.+a Magnetohydrodynamics and electrohydrodynamics

1 Introduction

The Rayleigh-Taylor instability, which occurs when a heavy fluid is supported by a lighter one, has great relevance in astrophysical, geophysical, controlled fusion, and industrial processes such as supernova explosions, controlled thermonuclear fusion experiments, and the implosion of inertial-confinement-fusion capsules. The nonlinear aspects of such an instability in dispersive media were investigated by Whitham [1], and Karpman [2], with an important emphasis on obtaining the nonlinear Schrödinger equation. Using inverse-scattering transforms, Zakharov and Shabat [3] obtained the solution of the nonlinear Schrödinger equation in (1+1)-dimensions. Their analysis reveals that smooth initial data lead to localized soliton solutions, and is related to the terminal state of the modulational instability.

Wave instabilities are probably the most remarkable physical phenomena that may occur in a nonlinear system [4]. Modulational instability and breakup of a continuous-wave field of large intensity was first predicted and analyzed in the context of waves in fluids [1]. A similar effect in the self-focusing of light in optical media with a nonlinear response [5] is responsible for the appearance of hot spots and associated optical damage in media irradiated by high power laser pulses. Self-focusing processes provide a wide area of investigations in various fields of physics: in optics, the light self-focusing results from the compressing influence of a Kerr type nonlinearity on the spatial profile of the beam [6–9]. In plasmas, the pondermotive filamentation of high energy laser pulses is initiated by slowly varying fluctuations of the electronic plasma density, and leads to beam fragmentation [10–13]. When self-focusing occurs in a medium, the wave field becomes singular at a point called the focus, and the amplitude of the wave packet becomes infinite, leading to turbulent bursts. This effect may be incorporated in studying the envelope properties in the (2+1)-dimensional wave packet [14].

Wave collapse, by which a singularity in the wave field is formed in a finite time, plays an important role in various branches of physics as one of the most effective mechanisms for the localization of the wave energy. One of the central problems in the collapse theory is to find

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the initial conditions collapse for which takes place. From a mathematical point of view such problems are related to the nonexistence theorems. These theorems determine the initial and/or boundary conditions for which the solution of the Cauchy problem only exists until some finite time [15]. Collapse (or blow-up) occurs when the amplitude of an unstable solitary wave localized in all dimensions grows to infinity in a finite time. As a matter of fact, the wave collapse is a particular scenario of the instability-induced evolution of a solitary wave under the action of perturbations of the same dimension [16–18]. Slunyaev et al. [19] have recently investigated the process of formation of huge waves on a finite water region as a result of dispersive wave grouping due to an appropriate phase modulation of the initial wave trains. For both one-, and two-dimensional surface movements, they have examined this process and compared it with the process of amplification due to modulational instability highlighting the importance of the optimal phase modulation, influence of nonlinear effects, and the presence of a random noise component in the wave field.

The nonlinear Schrödinger equation in its many versions is one of the most important models in mathematical physics, with applications in different fields such as plasma physics, water waves, bimolecular dynamics as well as nonlinear optics or quantum chemistry, to cite only a few cases (see e.g. Dodd et al. [20]. Hasegawa [8] and Sulem and Sulem [21,22], and the references therein). In many of those examples the equation appears as an asymptotic limit for a slowly varying dispersive wave envelope propagating in a nonlinear medium [23–29]. This equation, which is a generic equation modelling a variety of wave phenomena, plays a central role in the wave collapse theory. In the mathematical literature the development of a singularity in finite time is mostly referred to as "blow-up" of the solution. Since the growth of the amplitude is associated with a spatial contraction of the wave packet, the phrase "wave collapse" rather than "blow-up" is commonly used by physicists. Typically, however, blow-up occurs before the wave has contracted substantially, i.e. a narrow fast growing spike emerges on the slowly collapsing wave packet [30–34]. The work of Zakharov and Synakh [35], in the context of the self-focusing of optical beams in nonlinear media, has shown that under certain circumstances, the two-dimensional nonlinear Schrödinger equation develops a singularity and therefore focuses energy after a finite time. Consequently this behaviour shows that nonlinear focusing and therefore growth can set in, even in a linearly stable regime [36].

The subject of electrohydrodynamics (EHD) has drawn considerable interest over the past few decades, and it has a wide range of importance in various physical situations (see for example, Refs. [37–39], and references therein). Problems of nonlinear electrohydrodynamic stability have been considered by many authors in recent years. El-Sayed and Callebaut [40], in a series of papers, studied the slow modulation of the interfacial capillary-gravity waves of two superposed dielectric fluids of uniform depth under the influence of a general applied electric field (tangential or normal) to the interface between the two fluids, in the presence (or absence) of surface charges at their interfaces. It was shown that the complex amplitude of quasi-monochromatic travelling waves can be described by a nonlinear Schrödinger equation in a frame of reference moving with the group velocity. The stability characteristics of a uniform wave train are examined analytically and numerically on the basis of the nonlinear Schrödinger equation. Also, the complex amplitude of quasi-monochromatic standing waves near the cutoff wavenumber is found to be governed by a similar type of nonlinear Schrödinger equation in which the roles of time and space are interchanged. This equation makes it possible to estimate the nonlinear effect on the cutoff wavenumber. They [40] have also obtained the envelope solutions of the steady state in (2+1)-dimensions, in terms of the Jacobian elliptic functions. It follows that various types of envelope solutions of the modulated Stokes wave may exist depending on the relative sign of the terms representing the dispersive and nonlinear effects. The solitary and periodic envelope solutions for the general case of any liquid depth are described. It is shown also that the evolution of the envelope is governed by two coupled partial differential equations with cubic nonlinearity. The stability of the solitons with respect to transverse perturbations was also investigated. It was found that such wave packets are stable to short waves, and the waveguides are always unstable, and that the stability of the system depends on the values of the dielectric constant ratio, the electric field, the wavenumber, and the depth of the fluid. Callebaut and El-Sayed [41] have investigated the nonlinear electrohydrodynamic stability of solitary wave packets of capillary-gravity waves in (2+1)-dimensions. They found that the complex amplitude of the surface elevation can be described by a nonlinear Schrödinger equation, which can be written in the form of a soliton envelope equation. They obtained, using the tanh method, in a very simple way, the solitary wave solutions of this equation, which had been obtained before by them [40], using the Jacobian elliptic functions. They [41] had also used a variation of the tanh method to obtain alternative kinds of solitary wave solutions of the extended mKdV-KdV-Burgers equation.

In this paper, we have investigated the nonlinear electrohydrodynamic wave propagation of two superposed dielectric fluids in the presence of a horizontal electric field using the multiple time scales method in (2+1)-dimensions. The two-dimensional nonlinear Schrödinger equation for the evolution of the amplitude of progressive waves has been obtained, and some classes of soliton solutions of this equation for both the elliptic and hyperbolic cases are introduced. The phenomenon of nonlinear wave collapse (or blow up) of the wave packets is studied analytically and numerically. Finally, the modulational instability for the corresponding one-dimensional nonlinear Schrödinger equation for both the travelling and standing waves are discussed in detail, and the corresponding solitary wave solutions, and instability conditions are obtained.

2 Problem formulation

We consider the finite amplitude three-dimensional wave propagation on the interface $z = 0$ which separates the two semi-infinite dielectric inviscid incompressible fluids. The fluid with density $\rho^{(1)}$, and dielectric constant $\varepsilon^{(1)}$ occupies the region $z < 0$, whereas the region $z > 0$ is occupied by the fluid of density $\rho^{(2)}$, and dielectric constant $\varepsilon^{(2)}$. The fluids are influenced by a constant electric field E_0 in the x-direction. We nondimensionalize the various quantities using the characteristic length $l = \sqrt{T/\rho^{(1)}g}$, and the characteristic time $\sqrt{l/g}$, where T is the coefficient of surface tension, and g is the acceleration due to gravity acting in the negative z-direction. The motion is assumed to be non-rotational, then there are velocity potentials $\Phi^{(1),(2)}$ such that $\mathbf{v}^{(1),(2)} = \nabla\Phi^{(1),(2)}$.

The basic equations governing the perturbed velocity potentials $\Phi^{(1),(2)}$ are

$$\Phi^{(1)}_{xx} + \Phi^{(1)}_{yy} + \Phi^{(1)}_{zz} = 0, \quad -\infty < z < \eta(x,y,t), \tag{1}$$

$$\Phi^{(2)}_{xx} + \Phi^{(2)}_{yy} + \Phi^{(2)}_{zz} = 0, \quad \eta(x,y,t) < z < \infty, \tag{2}$$

where $z = \eta(x,y,t)$ is the elevation of the free interface measured from the unperturbed level. We assume that the quasi-static approximation is valid, and the electric field $\mathbf{E}$ is non-rotational. The electric potentials $\Psi^{(1),(2)}$ are defined such that

$$\mathbf{E}^{(1),(2)} = E_0\mathbf{e}_x - \nabla\Psi^{(1),(2)}, \tag{3}$$

and

$$\Psi^{(1)}_{xx} + \Psi^{(1)}_{yy} + \Psi^{(1)}_{zz} = 0, \quad -\infty < z < \eta(x,y,t), \tag{4}$$

$$\Psi^{(2)}_{xx} + \Psi^{(2)}_{yy} + \Psi^{(2)}_{zz} = 0, \quad \eta(x,y,t) < z < \infty. \tag{5}$$

Since the motion must vanish away from the interface, we must have

$$|\nabla\Phi^{(1)}| \to 0, \; |\nabla\Psi^{(1)}| \to 0, \quad \text{as} \quad z \to -\infty, \tag{6}$$

$$|\nabla\Phi^{(2)}| \to 0, \; |\nabla\Psi^{(2)}| \to 0, \quad \text{as} \quad z \to \infty. \tag{7}$$

The boundary conditions at the interface $z = \eta(x,y,t)$ are

(i) The condition that the interface is moving with the fluid leads to

$$\eta_t - \Phi^{(1),(2)}_z + \eta_x\Phi^{(1),(2)}_x + \eta_y\Phi^{(1),(2)}_y = 0, \quad \text{at} \;\; z = \eta(x,y,t). \tag{8}$$

(ii) The tangential component of the electric field is continuous at the interface

The unit normal vector $\mathbf{N}$ to the interface, is given by

$$\mathbf{N} = \frac{-\eta_x e_x - \eta_y e_y + e_z}{\sqrt{1+\eta_x^2+\eta_y^2}} \tag{9}$$

and then condition (ii) leads to

$$\eta_y[\Psi_z] + [\Psi_y] = 0, \;\; \text{at} \quad z = \eta(x,y,t) \tag{10}$$

$$\eta_x[\Psi_z] + [\Psi_x] = 0, \;\; \text{at} \quad z = \eta(x,y,t) \tag{11}$$

$$\eta_x[\Psi_y] - \eta_y[\Psi_x] = 0, \;\; \text{at} \quad z = \eta(x,y,t) \tag{12}$$

where $[\![\cdot]\!]$ represents the jump across the interface.

(iii) The normal electric displacement is continuous at the interface, and hence we obtain

$$\eta_x[\![\varepsilon\Psi_x]\!] + \eta_y[\![\varepsilon\Psi_y]\!] - [\![\varepsilon\Psi_z]\!] = \eta_x E_0[\![\varepsilon]\!], \;\; \text{at} \quad z = \eta(x,y,t). \tag{13}$$

(iv) The normal hydrodynamic stress is balanced by the normal electric stress. The balance condition is then

$$\begin{aligned}
&\Phi^{(1)}_t - \rho\Phi^{(2)}_t + (1-\rho)\eta + \frac{1}{2}\left[\Phi^{(1)2}_x - \rho\Phi^{(2)2}_x\right] \\
&+ \frac{1}{2}\left[\Phi^{(1)2}_y - \rho\Phi^{(2)2}_y\right] + \frac{1}{2}\left[\Phi^{(1)2}_z - \rho\Phi^{(2)2}_z\right] = \\
&\left(1+\eta_x^2+\eta_y^2\right)^{-3/2}\left[\eta_{xx}(1+\eta_y^2) + \eta_{yy}(1+\eta_x^2)\right. \\
&\left.-2\eta_{xy}\eta_x\eta_y\right] - \frac{1}{2}\left([\![\varepsilon\Psi_x^2]\!] + [\![\varepsilon\Psi_y^2]\!] - [\![\varepsilon\Psi_z^2]\!]\right) + E_0[\![\varepsilon\Psi_x]\!] \\
&+ 2\eta_x E_0[\![\varepsilon\Psi_z]\!] - 2\eta_x\eta_y E_0[\![\varepsilon\Psi_y]\!] - 2\eta_x[\![\varepsilon\Psi_x\Psi_z]\!] - 2\eta_y[\![\varepsilon\Psi_y\Psi_z]\!] \\
&+ \eta_x^2 E_0^2[\![\varepsilon]\!] - 2\eta_x^2 E_0[\![\varepsilon\Psi_x]\!] + \text{higher order terms}, \\
&\text{at} \quad z = \eta(x,y,t).
\end{aligned} \tag{14}$$

Equation (8) represents the kinematic condition, and equations (10–13) represent the continuity of the tangential component of the electric field, and the normal electric displacement at the interface, respectively; while equation (14) represents the balance condition between the normal hydrodynamic stress and the normal electric stress.

In order to describe the nonlinear interactions of the small but finite amplitude waves, we use the derivative expansion method with multiple scales. Following the usual procedure [42], let us expand η, $\Phi^{(1),(2)}$, and $\Psi^{(1),(2)}$ in the following asymptotic series

$$\eta(x,y,t) = \sum_{n=1}^{N+1}\delta^n\eta_n(x_0,x_1,\ldots,x_N,y_0,y_1,\ldots,y_N,t_0,t_1,\ldots,t_N) + O\left(\varepsilon^{N+2}\right), \tag{15}$$

$$\left(\Phi^{(1),(2)},\Psi^{(1),(2)}\right)(x,y,t) = \sum_{n=1}^{N+1}\delta^n\left(\Phi^{(1),(2)}_n,\Psi^{(1),(2)}_n\right) \times (x_0,x_1,\ldots,x_N,y_0,y_1,\ldots,y_N,z,t_0,t_1,\ldots,t_N) + O\left(\varepsilon^{N+2}\right), \tag{16}$$

where δ is a small parameter indicating the weakness of the nonlinearity. As shall be dealing with modulational instability, it is clear that the characteristic growth time has to be much longer (at least an order of magnitude) than the period of oscillation. Note that the expansion of η in equation (15) is independent of z.

The multiple scales $x_n(\equiv \delta^n x)$, $y_n(\equiv \delta^n y)$, and $t_n(\equiv \delta^n t)$ are assumed to satisfy the following derivative expansions

$$\frac{\partial}{\partial\alpha} = \sum_{n=0}^{N+1} \delta^n \frac{\partial}{\partial\alpha_n} + O\left(\varepsilon^{N+2}\right), \tag{17}$$

where α is any of the variables x, y, and t. It turns out for the present problem that it is sufficient to take $N = 2$, so far as the lowest significant order of approximation is concerned.

Expanding now the boundary conditions (8) and (10–14) into Taylor series around the undisturbed surface $z = 0$, then substituting (15–17) into equations (1, 2, 4), and (5) and the boundary conditions (8) and (10–14), and equating the coefficients of the same powers in δ, we can obtain a sequence of sets of equations for η_n, $\Phi_n^{(1),(2)}$, and $\Psi_n^{(1),(2)}$, given in the Appendix.

3 The evolution of wave packets

We assume that there is no steady flow in the undisturbed state, so that we choose the following quasi-monochromatic waves as the starting solutions to the first order problem

$$\eta_1 = A\exp(i\theta) + \text{c.c.}, \tag{18}$$

$$\Phi_1^{(1)} = -\frac{i\omega}{K} A\exp(i\theta + Kz) + \text{c.c.}, \tag{19}$$

$$\Phi_1^{(2)} = \frac{i\omega}{K} A\exp(i\theta - Kz) + \text{c.c.}, \tag{20}$$

$$\Psi_1^{(1)} = \frac{iE_0(\varepsilon^{(2)} - \varepsilon^{(1)})k}{(\varepsilon^{(1)} + \varepsilon^{(2)})K} A\exp(i\theta + Kz) + \text{c.c.}, \tag{21}$$

$$\Psi_1^{(2)} = \frac{iE_0(\varepsilon^{(2)} - \varepsilon^{(1)})k}{(\varepsilon^{(1)} + \varepsilon^{(2)})K} A\exp(i\theta - Kz) + \text{c.c.}, \tag{22}$$

where $\theta = kx_0 + ly_0 - \omega t_0$ is the phase of the carrier wave, k and l are the wavenumber components in the x-, and y-directions, respectively, $K = \sqrt{k^2 + l^2}$, and ω being the frequency, and c.c. stands for the complex conjugate of the preceding term (or terms), and i is the imaginary unit. Here, the complex amplitude of the surface elevation A is a function of slow scales x_1, x_2, y_1, y_2, t_1, and t_2.

In order that the starting solution should not be trivial, the total wavenumber K, and the frequency ω must satisfy the following dispersion relation

$$\omega^2 = \frac{K}{1+\rho}\left[1 - \rho + K^2 + \frac{k^2 E_0^2(\varepsilon^{(2)} - \varepsilon^{(1)})^2}{(\varepsilon^{(1)} + \varepsilon^{(2)})K}\right]. \tag{23}$$

The dispersion relation (23) was initially obtained by Melcher [37], Mohamed and Elshehawey [43] (for the case of two-dimensional, semi-infinite fluids), and therefore their results are recovered, and show that both the surface tension and the tangential electric field have stabilizing effects.

To derive the equation for the evolution of travelling waves, we need to proceed to the second order, and higher order problems. Since our aim is to study the amplitude modulation for travelling waves when $\omega^2 > 0$, we shall substitute the starting solutions given by equations (18–22) into the right-hand sides of the second order equations. In order to obtain the condition that the second order solutions be free of singularities, we can write the solution of the second order problem in the form

$$\eta_2 = \Lambda A^2 \exp(2i\theta) + \text{c.c.}, \tag{24}$$

$$\Phi_2^{(1)} = \frac{1}{K}\left[\frac{\partial A}{\partial t_1} + \frac{\omega(1-Kz)}{K^2}\left\{k\frac{\partial A}{\partial x_1} + l\frac{\partial A}{\partial y_1}\right\}\right] \times \exp(i\theta + Kz) - \frac{i\omega}{K}(\Lambda - K)A^2 \times \exp 2(i\theta + Kz) + \text{c.c.}, \tag{25}$$

$$\Phi_2^{(2)} = -\frac{1}{K}\left[\frac{\partial A}{\partial t_1} + \frac{\omega(1+Kz)}{K^2}\left\{k\frac{\partial A}{\partial x_1} + l\frac{\partial A}{\partial y_1}\right\}\right] \times \exp(i\theta - Kz) + \frac{i\omega}{K}(\Lambda + K)A^2 \times \exp 2(i\theta - Kz) + \text{c.c.}, \tag{26}$$

$$\Psi_2^{(1)} = \frac{E_0(\varepsilon^{(2)} - \varepsilon^{(1)})}{(\varepsilon^{(1)} + \varepsilon^{(2)})K}\left[\frac{\partial A}{\partial x_1} - \frac{k(1-Kz)}{K^2} \times \left\{k\frac{\partial A}{\partial x_1} + l\frac{\partial A}{\partial y_1}\right\}\right]\exp(i\theta + Kz) + \frac{ikE_0(\varepsilon^{(2)} - \varepsilon^{(1)})}{(\varepsilon^{(1)} + \varepsilon^{(2)})K}(\Lambda - K)A^2 \times \exp 2(i\theta + Kz) + \text{c.c.}, \tag{27}$$

$$\Psi_2^{(2)} = \frac{E_0(\varepsilon^{(2)} - \varepsilon^{(1)})}{(\varepsilon^{(1)} + \varepsilon^{(2)})K}\left[\frac{\partial A}{\partial x_1} - \frac{k(1+Kz)}{K^2} \times \left\{k\frac{\partial A}{\partial x_1} + l\frac{\partial A}{\partial y_1}\right\}\right] \times \exp(i\theta - Kz) + \frac{ikE_0(\varepsilon^{(2)} - \varepsilon^{(1)})}{(\varepsilon^{(1)} + \varepsilon^{(2)})K}(\Lambda + K)A^2 \times \exp 2(i\theta - Kz) + \text{c.c.}, \tag{28}$$

where

$$\Lambda = \frac{K(1-\rho)}{(1+\rho)(1-\rho-2K^2)}\left[(1-\rho) + K^2 + \frac{k^2 E_0^2(\varepsilon^{(2)} - \varepsilon^{(1)})^2}{(\varepsilon^{(1)} + \varepsilon^{(2)})K}\left\{1 + \frac{(1+\rho)(\varepsilon^{(2)} - \varepsilon^{(1)})}{(1-\rho)(\varepsilon^{(1)} + \varepsilon^{(2)})}\right\}\right]. \tag{29}$$

The case when $(1-\rho=0)$ or $(1-\rho-2K^2=0)$, for which $\eta_2, \Phi_2^{(1),(2)}$, and $\Psi_2^{(1),(2)}$ become infinite, corresponds to the case of the second harmonic resonance which can be dealt with along the same lines outlined by Lee [44], among others. In this section, we have assumed this quantity to be different from zero in equations (24–28).

The non-secularity condition for the second order perturbation is obtained from the last boundary condition of the second order equations, by equating to zero the coefficient of $\exp(i\theta)$, to obtain

$$\frac{\partial A}{\partial t_1} + v_k \frac{\partial A}{\partial x_1} + v_l \frac{\partial A}{\partial y_1} = 0, \tag{30}$$

together with its c.c., where $v_k = d\omega/dk$ and $v_l = d\omega/dl$ are the group velocities of the wave train the x-, and y-directions, respectively. Equation (30) implies that, for the lowest order in δ, the complex amplitude A remains constant in a frame of reference moving with the group velocities, that is, A depends on x_1, y_1, and t_1 only through $\zeta = x_1 + y_1 - (v_k + v_l)t_1 = \delta[x + y - (v_k + v_l)t]$.

4 Nonlinear EHD wave collapse

We now proceed to the third order problem. On using the first and second order solutions, we simplify the right hand side of the third order equations, and after some straightforward reductions, we can express the uniformly valid particular solutions for η_3, $\Phi_3^{(1),(2)}$, and $\Psi_3^{(1),(2)}$ in the form

$$\eta_3 = (K^2/2)|A|^2 A \exp(i\theta) + \text{c.c.}, \tag{31}$$

$$\begin{aligned}\Phi_3^{(1)} = &\left[\frac{1}{K}\frac{\partial A}{\partial t_2} + 3i\omega\left(\Lambda - \frac{K}{3}\right)|A|^2 A + \frac{i\omega}{2K^3}\right.\\ &\times\left\{(Kz-1)\left(\frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial y_1^2}\right) - \frac{2(Kz-1)}{\omega}\right.\\ &\times\left(k\frac{\partial}{\partial x_1} + l\frac{\partial}{\partial y_1}\right)\frac{\partial A}{\partial t_1} + \frac{1}{K^2}(3 - 3Kz + K^2z^2)\\ &\times\left(k\frac{\partial}{\partial x_1} + l\frac{\partial}{\partial y_1}\right)\left(k\frac{\partial A}{\partial x_1} + l\frac{\partial A}{\partial y_1}\right) + 2i(Kz-1)\\ &\times\left.\left.\left(k\frac{\partial A}{\partial x_2} + l\frac{\partial A}{\partial y_2}\right)\right\}\right]\exp(i\theta + Kz)\\ &+ \frac{1}{2K}\left[(\Lambda - K)\frac{\partial A^2}{\partial t_1} + \frac{\omega}{K^2}\Big\{(1 - 2Kz)\Lambda\right.\\ &\left.+2K^2z\Big\}\left(k\frac{\partial A^2}{\partial x_1} + l\frac{\partial A^2}{\partial y_1}\right)\right]\exp 2(i\theta + Kz)\\ &+ 3i\omega\left(\Lambda - \frac{K}{2}\right)A^3\exp 3(i\theta + Kz) + \text{c.c.}\end{aligned} \tag{32}$$

$$\begin{aligned}\Phi_3^{(2)} = &\left[-\frac{1}{K}\frac{\partial A}{\partial t_2} + 3i\omega\left(\Lambda + \frac{K}{3}\right)|A|^2 A + \frac{i\omega}{2K^3}\right.\\ &\times\left\{(Kz+1)\left(\frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial y_1^2}\right) - \frac{2(Kz+1)}{\omega}\right.\\ &\times\left(k\frac{\partial}{\partial x_1} + l\frac{\partial}{\partial y_1}\right)\frac{\partial A}{\partial t_1} - \frac{1}{K^2}(3 + 3Kz + K^2z^2)\\ &\times\left(k\frac{\partial}{\partial x_1} + l\frac{\partial}{\partial y_1}\right)\left(k\frac{\partial A}{\partial x_1} + l\frac{\partial A}{\partial y_1}\right) + 2i(Kz+1)\\ &\times\left.\left.\left(k\frac{\partial A}{\partial x_2} + l\frac{\partial A}{\partial y_2}\right)\right\}\right]\exp(i\theta - Kz) - \frac{1}{2K}\\ &\times\left[(\Lambda + K)\frac{\partial A^2}{\partial t_1} + \frac{\omega}{K^2}\{(1 + 2Kz)\Lambda\right.\\ &\left.+2K^2z\}\left(k\frac{\partial A^2}{\partial x_1} + l\frac{\partial A^2}{\partial y_1}\right)\right]\exp 2(i\theta - Kz)\\ &+ 3i\omega\left(\Lambda + \frac{K}{2}\right)A^3\exp 3(i\theta - Kz) + \text{c.c.}\end{aligned} \tag{33}$$

$$\begin{aligned}\Psi_3^{(1)} = &\left[\frac{iE_0(\varepsilon^{(2)} - \varepsilon^{(1)})}{2(\varepsilon^{(1)} + \varepsilon^{(2)})K^3}\left\{k(1 - Kz)\left(\frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial y_1^2}\right)\right.\right.\\ &+2(1 - Kz)\left(k\frac{\partial}{\partial x_1} + l\frac{\partial}{\partial y_1}\right)\frac{\partial A}{\partial x_1}\\ &-\frac{k}{K^2}(3 - 3Kz + K^2z^2)\left(k\frac{\partial}{\partial x_1} + l\frac{\partial}{\partial y_1}\right)\\ &\times\left(k\frac{\partial A}{\partial x_1} + l\frac{\partial A}{\partial y_1}\right) + 2ik(1 - Kz)\\ &\times\left.\left(k\frac{\partial A}{\partial x_2} + l\frac{\partial A}{\partial y_2}\right)\right\} + \frac{ikE_0(\varepsilon^{(2)} - \varepsilon^{(1)})}{(\varepsilon^{(1)} + \varepsilon^{(2)})}\\ &\times\left\{K + \frac{(\varepsilon^{(2)} - 3\varepsilon^{(1)})\Lambda}{(\varepsilon^{(1)} + \varepsilon^{(2)})}\right\}|A|^2 A\\ &\left.+\frac{E_0(\varepsilon^{(2)} - \varepsilon^{(1)})}{(\varepsilon^{(1)} + \varepsilon^{(2)})K}\frac{\partial A}{\partial x_2}\right]\exp(i\theta + Kz)\\ &+ \frac{E_0(\varepsilon^{(2)} - \varepsilon^{(1)})}{2(\varepsilon^{(1)} + \varepsilon^{(2)})K}\left[(\Lambda - K)\frac{\partial A^2}{\partial x_1} - \frac{k}{K^2}\right.\\ &\left.\times\left\{(1 - 2Kz)\Lambda + 2K^2z\right\}\left(k\frac{\partial A^2}{\partial x_1} + l\frac{\partial A^2}{\partial y_1}\right)\right]\\ &\times\exp 2(i\theta + Kz) - \frac{3ikE_0(\varepsilon^{(2)} - \varepsilon^{(1)})}{(\varepsilon^{(1)} + \varepsilon^{(2)})}\left(\Lambda - \frac{K}{2}\right)\\ &\times A^3\exp 3(i\theta + Kz) + \text{c.c.}\end{aligned} \tag{34}$$

$$\Psi_3^{(2)} = \left[\frac{iE_0(\varepsilon^{(2)}-\varepsilon^{(1)})}{2(\varepsilon^{(1)}+\varepsilon^{(2)})K^3}\left\{k(1+Kz)\left(\frac{\partial^2 A}{\partial x_1^2}+\frac{\partial^2 A}{\partial y_1^2}\right)\right.\right.$$
$$+2(1+Kz)\left(k\frac{\partial}{\partial x_1}+l\frac{\partial}{\partial y_1}\right)\frac{\partial A}{\partial x_1}$$
$$-\frac{k}{K^2}(3+3Kz+K^2z^2)\left(k\frac{\partial}{\partial x_1}+l\frac{\partial}{\partial y_1}\right)$$
$$\times\left(k\frac{\partial A}{\partial x_1}+l\frac{\partial A}{\partial y_1}\right)+2ik(1+Kz)$$
$$\left.\times\left(k\frac{\partial A}{\partial x_2}+l\frac{\partial A}{\partial y_2}\right)\right\}+\frac{ikE_0(\varepsilon^{(2)}-\varepsilon^{(1)})}{(\varepsilon^{(1)}+\varepsilon^{(2)})}$$
$$\times\left\{K+\frac{(3\varepsilon^{(2)}-\varepsilon^{(1)})\Lambda}{(\varepsilon^{(1)}+\varepsilon^{(2)})}\right\}|A|^2A$$
$$\left.+\frac{E_0(\varepsilon^{(2)}-\varepsilon^{(1)})}{(\varepsilon^{(1)}+\varepsilon^{(2)})K}\frac{\partial A}{\partial x_2}\right]\exp(i\theta-Kz)$$
$$+\frac{E_0(\varepsilon^{(2)}-\varepsilon^{(1)})}{2(\varepsilon^{(1)}+\varepsilon^{(2)})K}\left[(\Lambda+K)\frac{\partial A^2}{\partial x_1}\right.$$
$$\left.-\frac{k}{K^2}\left\{(1+2Kz)\Lambda+2K^2z\right\}\left(k\frac{\partial A^2}{\partial x_1}+l\frac{\partial A^2}{\partial y_1}\right)\right]$$
$$\times\exp 2(i\theta-Kz)+\frac{3ikE_0(\varepsilon^{(2)}-\varepsilon^{(1)})}{(\varepsilon^{(1)}+\varepsilon^{(2)})}\left(\Lambda+\frac{K}{2}\right)$$
$$\times A^3\exp 3(i\theta-Kz)+\text{c.c.} \tag{35}$$

Finally, substituting from the third order solution into the last boundary condition of the third order equations, we obtain for the non-secularity condition from the coefficient of $\exp(i\theta)$, the following equation

$$2i\left\{\frac{\partial A}{\partial t_2}+v_k\frac{\partial A}{\partial x_2}+v_l\frac{\partial A}{\partial y_2}\right\}+P_1\frac{\partial^2 A}{\partial x_1^2}$$
$$+2P_2\frac{\partial^2 A}{\partial x_1\partial y_1}+P_3\frac{\partial^2 A}{\partial y_1^2}=Q|A|^2A, \tag{36}$$

where

$$P_1=-\frac{1}{\omega}v_k^2+\frac{K}{\omega(1+\rho)}\left\{\frac{(l^2-2k^2)(1-\rho+K^2)}{2K^4}\right.$$
$$\left.+\frac{E_0^2(\varepsilon^{(2)}-\varepsilon^{(1)})^3(l^2-k^2)}{(\varepsilon^{(1)}+\varepsilon^{(2)})K^3}+1\right\}+\frac{2k}{K^2}v_k \tag{37}$$

$$P_2=-\frac{1}{\omega}v_kv_l-\frac{kl}{2\omega K^2(1+\rho)}$$
$$\times\left\{3(1-\rho+K^2)+\frac{2KE_0^2(\varepsilon^{(2)}-\varepsilon^{(1)})^2}{(\varepsilon^{(1)}+\varepsilon^{(2)})}\right\}+\frac{1}{K^2}(kv_l+lv_k) \tag{38}$$

$$P_3=-\frac{1}{\omega}v_l^2+\frac{K}{\omega(1+\rho)}$$
$$\times\left\{\frac{(k^2-2l^2)(1-\rho+K^2)}{2K^4}+1\right\}+\frac{2l}{K^2}v_l \tag{39}$$

$$Q=\frac{K}{\omega(1+\rho)}\left[\left\{2\omega^2(1-\rho)+\frac{2k^2E_0^2(\varepsilon^{(2)}-\varepsilon^{(1)})^3}{(\varepsilon^{(1)}+\varepsilon^{(2)})^2}\right\}\Lambda\right.$$
$$\left.+\frac{K^2}{2}\left\{4(1-\rho)+K^2\right\}\right]. \tag{40}$$

The partial differential equation (36) is elliptic or hyperbolic, depending on the sign of $P=(P_2^2-P_1P_3)$. When P is negative, we have the elliptic case, for which the appropriate transformations are

$$\tau=t_2,\quad \Theta=x_2+y_2-(v_k+v_l)\,t_2 \tag{41}$$

and

$$\zeta_1=\frac{x_1}{\sqrt{P_1}},\quad \zeta_2=\left\{P_3-\frac{P_2^2}{P_1}\right\}^{-1/2}\left(y_1-\frac{P_2}{P_1}x_1\right). \tag{42}$$

Under such a transformation, equation (36) now reduced to a standard two-dimensional nonlinear Schrödinger equation. On rewriting equation (36) in the group velocity reference frame, we obtain the following elliptic equation

$$2i\frac{\partial A}{\partial\tau}+\frac{\partial^2 A}{\partial\zeta_1^2}+\frac{\partial^2 A}{\partial\zeta_2^2}=Q|A|^2A. \tag{43}$$

Equation (43) is the standard nonlinear Schrödinger equation in (2+1) dimensions. For the hyperbolic case, P should be positive. We introduce the transformations (41) with

$$\zeta_1=\frac{x_1}{\sqrt{P_1}},\quad \zeta_2=\left\{\frac{P_2^2}{P_1^2}-P_3\right\}^{-1/2}\left(\frac{P_2}{P_1}x_1-y_1\right). \tag{44}$$

Proceeding as before, we obtain

$$2i\frac{\partial A}{\partial\tau}+\frac{\partial^2 A}{\partial\zeta_1^2}-\frac{\partial^2 A}{\partial\zeta_2^2}=Q|A|^2A. \tag{45}$$

Equations (43) and (45) can be expressed in the form

$$2i\frac{\partial A}{\partial\tau}+\Delta_1\frac{\partial^2 A}{\partial\zeta_1^2}+\Delta_2\frac{\partial^2 A}{\partial\zeta_2^2}=Q|A|^2A, \tag{46}$$

with $\Delta_1=1$, and

$$\Delta_2=\ \ 1\quad\text{when}\quad P_2^2-P_1P_3<0 \tag{47}$$

$$\Delta_2=-1\quad\text{when}\quad P_2^2-P_1P_3>0. \tag{48}$$

It should be pointed out that in the small amplitude limit, solitary waves with damped oscillations may be viewed as envelope-soliton solutions of the nonlinear Schrödinger equation, such that the wave crests with the same speed as the envelope. This can be readily understood on physical grounds because, at an extremum of the linear phase speed, the group velocity is equal to the linear phase speed. Hence, close to the bifurcation point, it is possible for the speed of the carrier wave to match the group velocity at a wavenumber in the vicinity of the phase speed extremum.

Solitary wave envelopes are special solutions of equations (43) and (45), respectively, and can be written in the form

$$A = \pm a \text{ sech } \left\{ a\left(-\frac{Q}{2\gamma_{\pm}}\right)^{1/2} \Gamma \right\} \exp(i\lambda^{*}\tau) \quad (49)$$

with

$$\Gamma = \zeta_1 \cos\Phi + \zeta_2 \sin\Phi \quad (50)$$

$$\gamma_{\pm} = (1,\ \cos\Phi) \quad (51)$$

$$\lambda^{*} = (1/2)Qa^2 \quad (52)$$

where a is the peak amplitude of the packet. These solutions are valid only if $Q\gamma_{\pm} < 0$. It is clear that a solitary envelope moves with the group velocity $(v_k,\ v_l)$, and is inclined at an angle Φ to the x-axis, the direction of propagation of the wave crests.

Suppose that the solution of the elliptic equation (43) can be written in the form [13]

$$A = \Theta(\tau, \zeta_1, \zeta_2) \exp[i(c_0\tau + c_1\zeta_1 + c_2\zeta_2)] \quad (53)$$

where c_0, c_1, and c_2 are real constants. Substituting from equation (53) into equation (43), we obtain

$$2i\frac{\partial\Theta}{\partial\tau} + 2ic_1\frac{\partial\Theta}{\partial\zeta_1} + 2ic_2\frac{\partial\Theta}{\partial\zeta_2} + \frac{\partial^2\Theta}{\partial\zeta_1^2} + \frac{\partial^2\Theta}{\partial\zeta_2^2} = Q\Theta^3 - C\Theta \quad (54)$$

where we have put $2c_0 + c_1^2 + c_2^2 = -C$ (a constant). To obtain the exponential solution of equation (54), we use the following function transformation

$$\Theta = \sqrt{C/Q}\ \exp(\phi/2) \quad (55)$$

where $\phi = \phi(\tau, \zeta_1, \zeta_2)$ is a function of τ, ζ_1, and ζ_2, then equation (54) can be written in the form

$$2i\frac{\partial\phi}{\partial\tau} + ic_1\frac{\partial\phi}{\partial\zeta_1} + ic_2\frac{\partial\phi}{\partial\zeta_2} + \frac{\partial^2\phi}{\partial\zeta_1^2} + \frac{\partial^2\phi}{\partial\zeta_2^2} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial\zeta_1}\right)^2 + \left(\frac{\partial\phi}{\partial\zeta_2}\right)^2\right] = 2C[\exp(\phi) - 1]. \quad (56)$$

Setting $\phi = \phi(\zeta)$ as a function of a single parameter ζ, then equation (56) yields

$$\left[2i\frac{\partial\zeta}{\partial\tau} + 2ic_1\frac{\partial\zeta}{\partial\zeta_1} + 2ic_2\frac{\partial\zeta}{\partial\zeta_2} + \frac{\partial^2\zeta}{\partial\zeta_1^2} + \frac{\partial^2\zeta}{\partial\zeta_2^2}\right]\frac{d\phi}{d\zeta} + \left[\left(\frac{\partial\zeta}{\partial\zeta_1}\right)^2 + \left(\frac{\partial\zeta}{\partial\zeta_2}\right)^2\right]\left[\frac{d^2\phi}{d\zeta^2} + \frac{1}{2}\left(\frac{\partial\phi}{\partial\zeta}\right)^2\right] = 2C[\exp(\phi) - 1]. \quad (57)$$

Explicitly, some solutions of equation (57) obey the following system of equations

$$2i\frac{\partial\zeta}{\partial\tau} + 2ic_1\frac{\partial\zeta}{\partial\zeta_1} + 2ic_2\frac{\partial\zeta}{\partial\zeta_2} = \frac{\partial^2\zeta}{\partial\zeta_1^2} + \frac{\partial^2\zeta}{\partial\zeta_2^2} = 0 \quad (58)$$

$$\left(\frac{\partial\zeta}{\partial\zeta_1}\right)^2 + \left(\frac{\partial\zeta}{\partial\zeta_2}\right)^2 = 1 \quad (59)$$

$$\frac{d^2\phi}{d\zeta^2} + \frac{1}{2}\left(\frac{\partial\phi}{\partial\zeta}\right)^2 = 2C[\exp(\phi) - 1]. \quad (60)$$

The solution of equation (60) can be written in the form

$$\phi = 2\ln\left[\sqrt{2}\ \sec\left(\sqrt{C}\ \zeta + \zeta_0\right)\right], \text{ and } \zeta_0 = \text{const.} \quad (61)$$

Substituting from equation (61) into equation (55), the soliton solution (53) of the two-dimensional nonlinear Schrödinger equation (43) will be given by

$$A = \sqrt{2C/Q}\ \sec\left(\sqrt{C}\ \zeta + \zeta_0\right) \times \exp[i(c_0\tau + c_1\zeta_1 + c_2\zeta_2)]. \quad (62)$$

Note that, ζ denotes a solution of equations (58) and (59), and due to the fact that these two equations have many different solutions, then equation (62) should include many interesting solitons of the two-dimensional nonlinear Schrödinger equation (43). In a similar way, we can obtain the exponential solution of the hyperbolic equation (45) in the form

$$A = \sqrt{2K/Q}\ \sec\left(\sqrt{K}\ \zeta + \zeta_0\right) \exp[i(n_0\tau + n_1\zeta_1 + n_2\zeta_2)] \quad (63)$$

where n_0, n_1, and n_2 are real constants satisfying the condition $2n_0 + n_1^2 + n_2^2 = -K$ (a constant).

Another kind of solution known as the Sinh-Gordon solution can be obtained, following the same procedure of Khater et al. [13], for both the elliptic and hyperbolic cases given by equations (43) and (45), respectively, in the form

$$A = \sqrt{C/Q}\ \sinh\left\{2\tanh^{-1}\left[\exp\left(\sqrt{2C}\ \zeta + \zeta_0\right) - \pi\right]\right\} \times \exp[i(c_0\tau + c_1\zeta_1 + c_2\zeta_2)] \quad (64)$$

and

$$A = \sqrt{K/Q}\ \sinh\left\{2\tanh^{-1}\left[\exp\left(\sqrt{2K}\ \zeta + \zeta_0\right) - \pi\right]\right\} \times \exp[i(n_0\tau + n_1\zeta_1 + n_2\zeta_2)]. \quad (65)$$

We shall now discuss the stability of the solution of equation (36). Following Zakharov [34], we shall define the following two integrals of motion for the elliptic case ($P < 0$):

$$N = \iint |A|^2\, d\zeta_1 d\zeta_2 \quad (66)$$

$$H = \frac{1}{2}\iint \left\{\Delta_1 \left|\frac{\partial A}{\partial\zeta_1}\right|^2 + \Delta_2\left|\frac{\partial A}{\partial\zeta_2}\right|^2 + \frac{Q}{2}|A|^2\right\} d\zeta_1 d\zeta_2 \quad (67)$$

where N is the wave action and H is the Hamiltonian. The use of equations (66) and (67) in equation (46) yields the virial equation

$$\frac{\partial^2 I}{\partial \tau^2} = 4H. \tag{68}$$

Here I, the moment of inertia of the wave packet, is defined as

$$I = \iint (\zeta_1^2 + \zeta_2^2) |A|^2 \, d\zeta_1 d\zeta_2. \tag{69}$$

Equation (68) when integrated gives

$$\langle I \rangle = \frac{2\tau^2 H}{N} + C_1 \tau + C_2 \tag{70}$$

where the average value of the inertia is given by

$$\langle I \rangle = \frac{1}{N} \iint (\zeta_1^2 + \zeta_2^2) |A|^2 \, d\zeta_1 d\zeta_2. \tag{71}$$

The constants of integration C_1 and C_2 are to be determined from the initial conditions $C_1 = \partial \langle I \rangle / \partial \tau|_{\tau=0} = \acute{I}(0)$, and $C_2 = \langle I \rangle |_{\tau=0} = I(0)$. Note that the constant C_2 is always positive. The sign of H in equation (70) is now important, since it is the $(2\tau^2 H/N)$ term which dominates as τ increases. If $Q < 0$, then it is possible to have $H < 0$ for quite a broad class of initial data. If therefore $H < 0$, then the right-hand side of equation (70) can change sign after a finite value of τ. Since the integral on the left-hand side of equation (71) has a positive integrand, this behaviour implies the existence of a singularity of A after a definite time $\tau = \tau_0$, and the solution ceases to exist. Berkshire and Gibbon [45] have established a close analogy to Sundman's results on the collapse in the N-body problem in classical mechanics by considering the integral in equation (71) as the moment of inertia and went on to describe the singularity as $(\tau_0 - \tau)^{-1/2}$. Landman et al. [46] have used a perturbation analysis with respect to the space dimensions to construct singular solutions of the two-dimensional nonlinear Schrödinger equation with cubic nonlinearity. These solutions blow up at a rate $[\ln \ln[(\tau_0 - \tau)^{-1}]]/(\tau_0 - \tau)]^{1/2}$, in contrast to the behaviour in three-dimensions where there is no logarithmic correction. The behaviour of the sudden increase in A after some finite time (i.e. when $|A| \to \infty$ at the collapse point) has been checked numerically by Konno and Suzuki [9], and they confirmed these results. If now $Q > 0$, then no singularity is predicted, and defocusing occurs. If $\Delta_2 = -1$, then for this version of the nonlinear Schrödinger equation (45), no singularity is predicted. For focusing or collapse to occur in a finite time, we require that $\langle I \rangle$ tends to zero, implying thereby $Q < 0$. This demands that $|A| \to 0$ everywhere except at the focus. Total wave collapse will take place provided that the blow-up singularities do not occur at times earlier than the collapse by the virial theorem. Rasmussen and Rypdal [47], and Berge [11] have shown that the virial theorem cannot be used to predict the time of blow up, and it fails to give information about whether or not $\langle I \rangle$ is concentrated at the singular point. In an actual physical case, the dissipative effects will inhibit the earlier collapse from occurring. However, Berge [11] has shown that collapse will not take place in the hyperbolic two-dimensional case. It is interesting to note here that the corresponding equation (67) for the Hamiltonian H does not contain any nonlinear term for the hyperbolic case. We should like to point out here that for collapse to occur in a finite time, the group velocity rate P, and the nonlinear interaction coefficient Q must change sign as the quantities ρ, k, l, $\varepsilon(= \varepsilon^{(1)}/\varepsilon^{(1)})$, and $V = \varepsilon^{(1)} E_0^2$ are varied. Moreover, part of the region which is stable in linear theory becomes unstable because of the nonlinear focusing. The transition occurs from a marginal state to an excited one for the subcritical value of the applied electric field. Similar results have been obtained by Singh et al. [48] in their study of nonlinear dispersive instabilities in Kelvin-Helmholtz magnetohydrodynamic flows. They have shown that, when the velocity difference U is less than a critical velocity U_c, then the equation governing the amplitude evolves into a self-focusing singularity, and that the self-focusing of waves which predominates, at short wavelengths, is directionally dependent. They have also shown numerically that the self-focusing depends sensitively on the strength of the applied magnetic field, and the minimum velocity that allows the existence of self-focusing increases with increasing magnetic field strength, and also that this phenomena takes place for the elliptic case only.

It is clear that the above results reveal the existence of a region for nonlinear focusing when the wavenumbers $K^2 > K_c^2 = (1/2)(1 - \rho)$. The focusing condition is sensitive to the electric field velocity $V(= \varepsilon^{(1)} E_0^2)$, and the critical wavenumbers at which this condition is satisfied increases with increasing electric field strength and density ratio. Moreover, the focusing has directional dependence, and is significant at shorter wavelengths. The important mechanism therefore for growth in A after a finite time holds only when $\Delta_2 = 1$, and $Q < 0$. In particular, if $C_1(0) = 0$, then the collapse occurs at time $\tau_0 = \sqrt{C_2/4H}$ at a point in the (ζ_1, ζ_2) plane. When $H > 0$, an additional forcing of the initial data is required for the collapse to occur, such as $C_1(0) < 0$ and $C_1^2(0) \geq 16HC_2(0)$. In the second case, following Berkshire and Gibbon [45], the conditions for collapse to occur in a finite time are $I > 0$, $C_1(0) < 0$, $C_2(0) > 0$, and the angular momentum is zero. Similar conditions have been obtained by Zubarev [38] in studying the formation of root singularities on the free surface of a conducting ideal fluid in a strong electric field. He found that the nonlinear equation for two-dimensional fluid motion can be solved with the small angle approximation. In this case, he showed that for almost any arbitrary initial conditions, the surface curvature becomes infinite in a finite time. Figures 1–5 depict three dimensional plots for the electric field parameter V versus the wavenumbers k and l at which such a self-focusing phenomenon takes place for the elliptic case only when the dielectric constant parameter $\varepsilon = \varepsilon^{(2)}/\varepsilon^{(1)} = 0.5$. Figures 1–3 are drawn for the density parameter $\rho = 0.2$ when the wavenumber values k, l: $0.1 \to (5, 7, 10)$, respectively, and show that the wave collapse occurs for wavenumbers values $k \leq 2$ and $l \geq 3$, and increases by

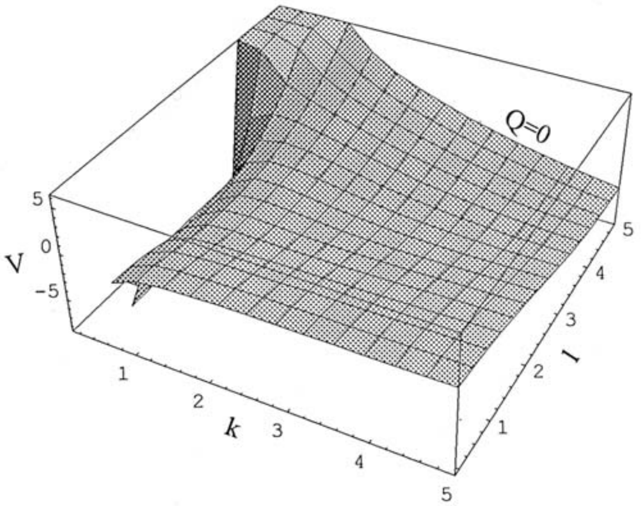

Fig. 1. A plot of the electric field parameter $V = \varepsilon^{(1)}E_0^2$ against the wavenumbers k and l in which k: $0.1 \rightarrow 5$ and l: $0.1 \rightarrow 5$ where $\rho = \rho^{(2)}/\rho^{(1)} = 0.2$ and $\varepsilon = \varepsilon^{(2)}/\varepsilon^{(1)} = 0.5$.

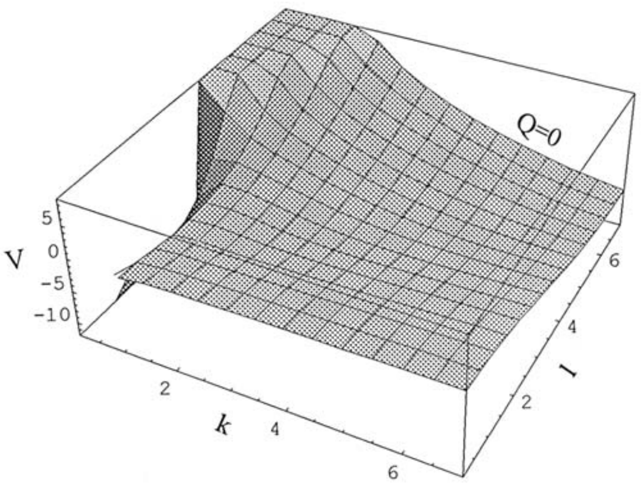

Fig. 2. A plot for the same system considered in Figure 1, but with k: $0.1 \rightarrow 7$ and l: $0.1 \rightarrow 7$.

increasing the electric field parameter V. Figures 4 and 5 are drawn for the density parameter $\rho = 1.2$ when the wavenumber values k, l: $(0.67, 0.7) \rightarrow (1, 10)$, respectively, and show that the self-focusing phenomenon holds for all values of the electric field parameter V when $k, l \geq 0.64$, and increases by increasing the applied electric field. Thus, the electric field increases the collapsing region after the critical wavenumbers k and l, which are different for the different cases when $\rho \lesseqgtr 1$, respectively. Note that in the second case when $\rho > 1$, the wave collapse occurs earlier than the first case when $\rho < 1$, and for small critical wavenumbers.

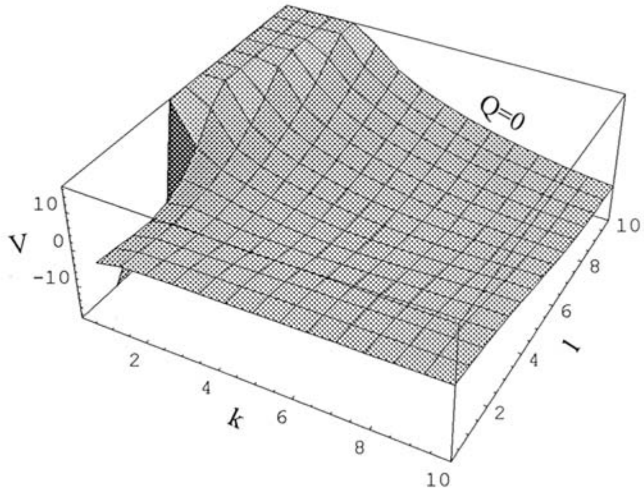

Fig. 3. A plot for the same system considered in Figure 1, but with k: $0.1 \rightarrow 10$ and l: $0.1 \rightarrow 10$.

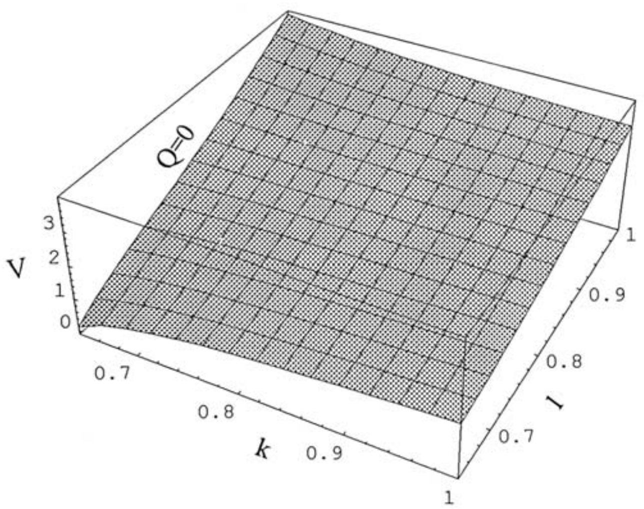

Fig. 4. A plot of the electric field parameter $V = \varepsilon^{(1)}E_0^2$ against the wavenumbers k and l in which k: $0.64 \rightarrow 1$ and l: $0.64 \rightarrow 1$ where $\rho = \rho^{(2)}/\rho^{(1)} = 1.2$ and $\varepsilon = \varepsilon^{(2)}/\varepsilon^{(1)} = 0.5$.

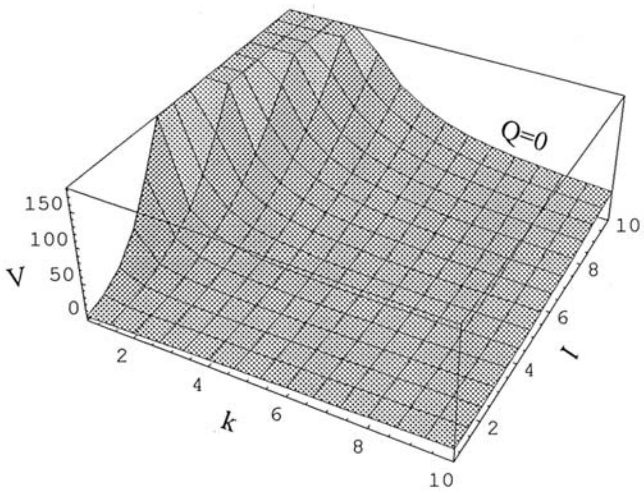

Fig. 5. A plot for the same system considered in Figure 4, but with k: $0.7 \rightarrow 10$ and l: $0.7 \rightarrow 10$.

5 Modulational instability

To investigate modulational instability for both the travelling and standing waves, we use the transformation

$$x_2 = x_1 - v_k t, \ \ y_2 = y_1 - v_l t, \ \text{ and } \ t_2 = t \tag{72}$$

then equation (36) takes the form

$$i\frac{\partial A}{\partial t} + \frac{1}{2}\left[P_1\frac{\partial^2 A}{\partial x_1^2} + 2P_2\frac{\partial^2 A}{\partial x_1 \partial y_1} + P_3\frac{\partial^2 A}{\partial y_1^2}\right] = Q_1|A|^2 A, \tag{73}$$

where $Q_1 = Q/2$. Equation (73) can be reduced to the one-dimensional nonlinear Schrödinger equation by the substitution [49]

$$X = mx + ny \quad \text{and} \quad T = t. \tag{74}$$

Thus, for the travelling waves, we have

$$i\frac{\partial A}{\partial T} + P\frac{\partial^2 A}{\partial X^2} = Q_1|A|^2 A, \tag{75}$$

where m and n are arbitrary constants, and

$$P = (1/2)\left(m^2 P_1 + 2mnP_2 + n^2 P_3\right). \tag{76}$$

It is also known that the solutions of equation (75) are unstable under modulation if $PQ_1 < 0$. The above equation (75) can also be written in the form

$$i\frac{\partial A}{\partial T} + P\frac{\partial^2 A}{\partial X^2} + \widetilde{Q}_1|A|^2 A = 0 \tag{77}$$

where $\widetilde{Q}_1 = -Q/2$. As is well known, when the nonlinear effects are small, the system of equations that describe the physical phenomena admit harmonic wave solutions with constant amplitude. If the amplitude of the wave is small-but-finite, the nonlinear terms can not be neglected and the nonlinearity gives rise to the variation of amplitude both in the space and time variables. When the amplitude varies slowly over a period of oscillation, a stretching transformation allows us to decompose the system into a rapidly varying part associated with the oscillation and slowly varying part such as the amplitude. A formal solution can be given in the form of an asymptotic expansion, and an equation determining the modulation of the first order amplitude can be derived. For instance, the nonlinear Schrödinger equation (77) is the simplest representative equation describing the self-modulation of one-dimensional monochromatic plane waves in dispersive media. It exhibits a balance between the nonlinearity and dispersion. We shall try here to present a progressive wave solution to the nonlinear Schrödinger equation (77). For that purpose, we shall propose a solution of the form [50]

$$A = a(T)V(\xi)\exp\{i[\Omega(T) - \kappa X]\}, \quad \xi = \alpha(T)(X - 2\kappa PT) \tag{78}$$

where κ is a constant, $a(T)$, $\alpha(T)$, and $V(\xi)$ are some real functions to be determined from the solution of equation (77). Introducing equation (78) into equation (77), and setting the real and imaginary parts of the resulting equation equal to zero, we obtain the following sets of ordinary differential equations

$$\frac{a'(T)}{a(T)}V + \frac{\alpha'(T)}{\alpha(T)}\zeta V' = 0 \tag{79}$$

$$\left[-\Omega'(T) - P\kappa^2\right]V + P\alpha^2(T)V'' + \widetilde{Q}_1 a^2(T)V^3 = 0 \tag{80}$$

where primes denote the derivative of the corresponding quantities with respect to its argument. Here, we shall be concerned with the localized travelling wave solution to the field equation, i.e. V and its various order derivatives vanish as $\xi \to \pm\infty$. Under these assumptions, one can show that the square of V is square integrable over the interval $(-\infty, \infty)$. Thus, multiplying equation (79) by V' and integrating the resulting equation with respect to ξ from $-\infty$ to ∞, we obtain

$$\left[\frac{a'(T)}{a(T)} - \frac{1}{2}\frac{\alpha'(T)}{\alpha(T)}\right]\left\langle V^2\right\rangle = 0,$$

$$\text{where} \quad \left\langle V^2\right\rangle = \int_{-\infty}^{\infty} V^2 d\xi. \tag{81}$$

Since $\langle V^2\rangle$ is bounded and non-zero, we obtain the following differential equation

$$\frac{a'(T)}{a(T)} - \frac{1}{2}\frac{\alpha'(T)}{\alpha(T)} = 0. \tag{82}$$

Now, in the first place, we shall propose a solution of equation (80) of the following form

$$V(\zeta)) = \text{sech}\ \xi. \tag{83}$$

Introducing equation (83) into equation (80), we obtain the following differential equations

$$\alpha^2(T) = \frac{\widetilde{Q}_1}{2P}a^2(T), \ \text{ and } \ \ \Omega'(T) = \frac{\widetilde{Q}_1}{2}a^2(T) - P\kappa^2. \tag{84}$$

It is seen from the first term of equation (84) that, in order to have a real solution $\alpha(T)$ and $a(T)$, the coefficients P and $\widetilde{Q}_1$ must satisfy the inequality $P\widetilde{Q}_1 > 0$. Now, let us return to the investigation of equation (79). It can be shown from the first term of equation (84) that $\alpha'(T)/\alpha(T) = a'(T)/a(T)$. Introducing this relation into equation (79), we obtain

$$\frac{a'(T)}{a(T)} = 0. \tag{85}$$

The solution of this equation may be given as follows

$$a(T) = a_0 \tag{86}$$

where a_0 is a constant. Introducing equation (86) into equation (84), we obtain

$$\alpha(T) = \left(\frac{\widetilde{Q}_1}{2P}\right)^{1/2} a_0 \quad \text{and} \quad \Omega(T) = \left(-P\kappa^2 + \frac{\widetilde{Q}_1 a_0^2}{2}\right)T. \tag{87}$$

Here, in obtaining the function $\Omega(T)$, we have utilized the regularity condition that $\Omega(T) = 0$ at $T = 0$. Thus the final solution may be given in the form

$$A = a_0 \operatorname{sech} \xi \exp\{i[\Omega(T) - \kappa X]\},$$
$$\xi = \left(\frac{\widetilde{Q}_1}{2P}\right)^{1/2} a_0 (X - 2\kappa P T). \tag{88}$$

Now, secondly, we shall propose a solution to equation (80) of the form

$$V = \tanh \xi. \tag{89}$$

Since this function is not square integrable, we cannot follow the above procedure to handle this problem. As is seen from equation (79), for this type of solution $\xi V'(\xi)$ approaches to zero as $\xi \to \infty$. Therefore, if we investigate the far field behaviour of equation (79) and consider that $\xi V'(\xi)$ vanishes as $\xi \to \infty$, we obtain the following differential equation

$$\frac{a'(T)}{a(T)} = 0. \tag{90}$$

The solution of this equation may be given by

$$a(T) = a_0 \tag{91}$$

where a_0 is a constant. Now, let us introduce the proposed solution (89) into equation (80). In order to satisfy this equation, the following relations must be satisfied

$$2P\alpha^2(T) + \widetilde{Q}_1 a_0^2 = 0 \tag{92}$$
$$-\Omega'(T) - P\kappa^2 + \widetilde{Q}_1 a_0^2 = 0. \tag{93}$$

The relation (92) holds true if and only if P and $\widetilde{Q}_1$ have different signs. Equation (92) also gives

$$\alpha(T) = \left(-\frac{\widetilde{Q}_1}{2P}\right)^{1/2} a_0. \tag{94}$$

Finally, integrating equation (93), and utilizing the regularity condition $\Omega(T) = 0$ at $T = 0$, we obtain

$$\Omega(T) = \left(-P\kappa^2 + \widetilde{Q}_1 a_0^2\right) T. \tag{95}$$

Thus, the final solution may be given by

$$A = a_0 \tanh \xi \exp\{i[\Omega(T) - \kappa X]\},$$
$$\xi = \left(-\frac{\widetilde{Q}_1}{2P}\right)^{1/2} a_0 (X - 2\kappa P T). \tag{96}$$

On the other hand, for the standing waves, we obtain the following one-dimensional nonlinear Schrödinger equation

$$i\left\{\frac{\partial A}{\partial x} + v_k \frac{\partial A}{\partial t}\right\} + \widehat{P}\frac{\partial^2 A}{\partial t^2} = \widehat{Q}|A|^2 A, \tag{97}$$

in which the roles of space and time are interchanged. Changing the independent variables from x and t to $\widehat{\xi} = t - v_k x$ and $\widehat{\eta} = x$, we can express equation (97) in the form

$$i\frac{\partial A}{\partial \widehat{\eta}} + \widehat{P}\frac{\partial^2 A}{\partial \widehat{\xi}^2} = \widehat{Q}|A|^2 A, \tag{98}$$

which is also a nonlinear Schrödinger equation [51].

Letting $A = (1/2)\widehat{a}\exp(i\widehat{\beta})$, with real functions $\widehat{a}$ and $\widehat{\beta}$, in equation (98) and separating the real and imaginary parts, we obtain

$$\frac{\partial \widehat{a}}{\partial \widehat{\eta}} + 2\widehat{P}\left[\frac{\partial \widehat{a}}{\partial \widehat{\xi}}\frac{\partial \widehat{\beta}}{\partial \widehat{\xi}} + \frac{\widehat{a}}{2}\frac{\partial^2 \widehat{\beta}}{\partial \widehat{\xi}^2}\right] = 0 \tag{99}$$
$$\frac{\partial \widehat{\beta}}{\partial \widehat{\eta}} - \widehat{P}\left[\frac{1}{\widehat{a}}\frac{\partial^2 \widehat{a}}{\partial \widehat{\xi}^2} - \left(\frac{\partial \widehat{\beta}}{\partial \widehat{\xi}}\right)^2\right] = -\frac{1}{4}\widehat{Q}\widehat{a}^2. \tag{100}$$

For monochromatic waves (a single frequency wave) the amplitude and phase are independent of t, so that $\partial\widehat{a}/\partial\widehat{\xi} = \partial\widehat{\beta}/\partial\widehat{\xi} = 0$, and equations (99) and (100) can be integrated to give

$$\widehat{a} = \widehat{a}_0 \quad \text{and} \quad \widehat{\beta} = -(1/4)\widehat{Q}\widehat{a}_0^2 + \widehat{\beta}_0 \tag{101}$$

where $\widehat{a}_0$ and $\widehat{\beta}_0$ are constants. Equations (99) and (100) can be used to analyze the stability of the corresponding monochromatic solution. To do this, we let

$$\widehat{a} = \widehat{a}_0 + \widehat{a}_1 \quad \text{and} \quad \widehat{\beta} = -(1/4)\widehat{Q}\widehat{a}_0^2\widehat{\eta} + \widehat{\beta}_0 + \widehat{\beta}_1 \tag{102}$$

where $\widehat{a}_1$ and $\widehat{\beta}_1$ are small compared with the preceding terms. Substituting equation (102) into equations (99) and (100), and neglecting the nonlinear terms in $\widehat{a}_1$ and $\widehat{\beta}_1$, we obtain

$$\frac{\partial \widehat{a}_1}{\partial \eta} + \widehat{P}\widehat{a}_0\frac{\partial^2 \widehat{\beta}_1}{\partial \xi^2} = 0 \tag{103}$$
$$\frac{\partial \widehat{\beta}_1}{\partial \eta} - \frac{\widehat{P}}{\widehat{a}_0}\frac{\partial^2 \widehat{a}_1}{\partial \xi^2} = -\frac{1}{2}\widehat{Q}\widehat{a}_0\widehat{a}_1. \tag{104}$$

Since equations (103) and (104) are linear, we seek their solutions in the form

$$\left(\widehat{a}_1, \widehat{\beta}_1\right) = \left(\widetilde{a}_1, \widetilde{\beta}_1\right)\exp\left[i\left(\widetilde{k}\widehat{\eta} - \widetilde{\omega}\widehat{\xi}\right)\right] \tag{105}$$

where $\widetilde{a}_1$ and $\widetilde{\beta}_1$ are constants. Substituting these solutions into equations (103) and (104), we obtain

$$\widetilde{k}^2 = \widetilde{\omega}^2\widehat{P}^2\left[\widetilde{\omega}^2 + \frac{\widehat{Q}\widehat{a}_0^2}{2\widehat{P}}\right] \tag{106}$$

which shows that, if $\widehat{Q}/\widehat{P} > 0$ (i.e. if either the conditions

$\widehat{P} > 0$ and $\widehat{Q} > 0$, or $\widehat{P} < 0$ and $\widehat{Q} < 0$ hold simultaneously), then $\widetilde{k}$ is always real for all values of $\widetilde{\omega}$ so that the monochromatic waves in this case are neutrally stable. On the other hand, if $\widehat{Q}/\widehat{P} < 0$ (i.e. if either the conditions $\widehat{P} > 0$ and $\widehat{Q} < 0$, or $\widehat{P} < 0$ and $\widehat{Q} > 0$ hold simultaneously), then $\widetilde{k}^2$ is negative for all $\widetilde{\omega} < (-\widehat{Q}\widehat{a}_0^2/2\widehat{P})^{1/2}$. Consequently, disturbances grow exponentially with $\widehat{\xi}$, and the monochromatic waves in this case are unstable. Also if $\widehat{Q}/\widehat{P} < 0$ and $\widetilde{\omega} > (-\widehat{Q}\widehat{a}_0^2/2\widehat{P})^{1/2}$ are satisfied, then the monochromatic waves are neutrally stable.

The modulation instability is known to be an effective physical mechanism in optics and fluids for the break up of continuous modes into solitary waves. Three kinds of modulation instabilities, i.e. temporal, spatial, and spatiotemporal instabilities [52] are known to occur. So far, theoretical analyses of the onset of spatiotemporal instability have been based on the standard (3+1)-dimensional nonlinear Schrödinger equation which results from Maxwell's and fluid equations under the assumption of a slowly varying envelope approximation. Such spatiotemporal instability will be presented in a separate subsequent paper in the near future.

6 Conclusions

In the present work, a weakly nonlinear theory of the electrohydrodynamic Rayleigh-Taylor configuration of two superposed dielectric fluids in the presence of a horizontal applied electric field in (2+1)-dimensions is investigated using the multiple time scales method. The basic equations and the boundary conditions appropriate to the problem are given leading to a sequence of sets of equations connecting different parameters. We have obtained the standard two-dimensional nonlinear Schrödinger equation in elliptic and hyperbolic forms. We have also obtained the general soliton solutions to this equation in both cases. This evolution equation of the wave packets in (2+1)-dimensions has been transformed, using the function transformation method, into an exponential and a Sinh-Gordon equation, and the corresponding solutions of these equations are obtained. The most dramatic consequence for the wave packets, i.e. nonlinear focusing or wave collapse is investigated. This phenomenon provides a local region in which amplitudes become unbounded in a finite time. The wave packet collapse is analyzed both analytically and numerically. It is found that the electric field increases the collapsing region for different cases when $\rho \lesseqgtr 1$, and that the wave collapse occurs earlier in the second case when $\rho > 1$, for some critical wavenumbers. The modulational instability for the corresponding one-dimensional nonlinear Schrödinger equation is obtained and discussed for both the travelling and standing waves. For the former case, it is found that the obtained evolution equation admits solitary wave solutions with variable wave amplitude and speed. For the later case, the resulting evolution equation has been used to show that the monochromatic waves are stable in this case, and to determine the amplitude dependence of the cutoff frequencies.

Appendix

Introducing the linear operators defined as

$$L\left[\Phi_n^{(1),(2)},\Psi_n^{(1),(2)}\right] = \left(\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial y_0^2}+\frac{\partial^2}{\partial z^2}\right) \times \left(\Phi_n^{(1),(2)},\Psi_n^{(1),(2)}\right) \tag{A.1}$$

$$M\left[\eta_n,\Phi_n^{(1),(2)}\right] = \frac{\partial\eta_n}{\partial t_0} - \frac{\partial\phi_n^{(1),(2)}}{\partial z} \tag{A.2}$$

$$O_{1,2}[\Psi_n] = \left[\!\!\left[\frac{\partial\Psi_n}{\partial(x_0,y_0)}\right]\!\!\right] \tag{A.3}$$

$$G[\eta_n,\Psi_n] = \left[\!\!\left[\varepsilon\frac{\partial\Psi_n}{\partial z}\right]\!\!\right] + \frac{\partial\eta_n}{\partial x_0}E_0\,[\varepsilon] \tag{A.4}$$

$$N[\eta_n,\Phi_n,\Psi_n] = \frac{\partial\Phi_n^{(1)}}{\partial t_0} - \rho\frac{\partial\Phi_n^{(2)}}{\partial t_0} + (1-\rho)\eta_n - frac\partial^2\eta_n\partial x_0^2 - \frac{\partial^2\eta_n}{\partial y_0^2} - E_0\left[\!\!\left[\varepsilon\left(\frac{\partial\Psi_n}{\partial x_0}\right)\right]\!\!\right]. \tag{A.5}$$

The first order problem for $O(\delta)$ is given by

$$L\left[\Phi_1^{(1),(2)},\Psi_1^{(1),(2)}\right] = 0, \tag{A.6}$$

$$M\left[\eta_1,\Phi_1^{(1),(2)}\right] = 0, \quad \text{at} \quad z=0 \tag{A.7}$$

$$O_{1,2}[\Psi_1] = 0, \quad \text{at} \quad z=0 \tag{A.8}$$

$$G[\eta_1,\Psi_1] = 0, \quad \text{at} \quad z=0 \tag{A.9}$$

$$N[\eta_1,\Phi_1,\Psi_1] = 0, \quad \text{at} \quad z=0. \tag{A.10}$$

The second order problem for $O(\delta^2)$ is given by

$$L\left[\Phi_2^{(1),(2)},\Psi_2^{(1),(2)}\right] = -2\left[\frac{\partial^2}{\partial x_0\partial x_1}+\frac{\partial^2}{\partial y_0\partial y_1}\right] \times \left(\Phi_1^{(1),(2)},\Psi_1^{(1),(2)}\right) \tag{A.11}$$

$$M\left[\eta_2,\Phi_2^{(1),(2)}\right] = -\frac{\partial\eta_1}{\partial t_1} + \eta_1\frac{\partial^2\Phi_1^{(1),(2)}}{\partial z^2} - \frac{\partial\eta_1}{\partial x_0}\frac{\partial\Phi_1^{(1),(2)}}{\partial x_0} - \frac{\partial\eta_1}{\partial y_0}\frac{\partial\Phi_1^{(1),(2)}}{\partial y_0}, \quad \text{at} \quad z=0 \tag{A.12}$$

$$O_{1,2}[\Psi_2] = -\left[\!\!\left[\frac{\partial\Psi_1}{\partial(x_1,y_1)}\right]\!\!\right] - \frac{\partial\eta_1}{\partial(x_0,y_0)}\left[\!\!\left[\frac{\partial\Psi_1}{\partial z}\right]\!\!\right] - \eta_1\left[\!\!\left[\frac{\partial^2\Psi_1}{\partial(x_0,y_0)\partial z}\right]\!\!\right], \quad \text{at} \quad z=0 \tag{A.13}$$

$$G[\eta_2,\Psi_2]=\frac{\partial\eta_1}{\partial x_0}\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial x_0}\right]\!\right]+\frac{\partial\eta_1}{\partial y_0}\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial y_0}\right]\!\right]-\eta_1\left[\!\left[\varepsilon\frac{\partial^2\Psi_1}{\partial z^2}\right]\!\right]-\frac{\partial\eta_1}{\partial x_1}E_0\left[[\varepsilon]\right],\ \text{at}\quad z=0 \tag{A.14}$$

$$\begin{aligned}N[\eta_2,\Phi_2,\Psi_2]=&-\left[\frac{\partial\Phi_1^{(1)}}{\partial t_1}-\rho\frac{\partial\Phi_1^{(2)}}{\partial t_1}\right]-\eta_1\left[\frac{\partial^2\Phi_1^{(1)}}{\partial t_0\partial z}-\rho\frac{\partial^2\Phi_1^{(2)}}{\partial t_0\partial z}\right]\\&-\frac{1}{2}\left[\left\{\frac{\partial\Phi_1^{(1)}}{\partial x_0}\right\}^2-\rho\left\{\frac{\partial\Phi_1^{(2)}}{\partial x_0}\right\}^2\right]-\frac{1}{2}\left[\left\{\frac{\partial\Phi_1^{(1)}}{\partial y_0}\right\}^2-\rho\left\{\frac{\partial\Phi_1^{(2)}}{\partial y_0}\right\}^2\right]\\&-\frac{1}{2}\left[\left\{\frac{\partial\Phi_1^{(1)}}{\partial z}\right\}^2-\rho\left\{\frac{\partial\Phi_1^{(2)}}{\partial z}\right\}^2\right]+2\frac{\partial^2\eta_1}{\partial x_0\partial x_1}+2\frac{\partial^2\eta_1}{\partial y_0\partial y_1}-\frac{1}{2}\left[\!\left[\varepsilon\left(\frac{\partial\Psi_1}{\partial x_0}\right)^2\right]\!\right]\\&-\frac{1}{2}\left[\!\left[\varepsilon\left(\frac{\partial\Psi_1}{\partial y_0}\right)^2\right]\!\right]+\frac{1}{2}\left[\!\left[\varepsilon\left(\frac{\partial\Psi_1}{\partial z}\right)^2\right]\!\right]+\eta_1E_0\left[\!\left[\varepsilon\left(\frac{\partial^2\Psi_1}{\partial x_0\partial z}\right)\right]\!\right]+E_0\left[\!\left[\varepsilon\left(\frac{\partial\Psi_1}{\partial x_1}\right)\right]\!\right]\\&+2\frac{\partial\eta_1}{\partial x_0}E_0\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial z}\right]\!\right]+\left(\frac{\partial\eta_1}{\partial x_0}\right)^2E_0^2\left[\!\left[\varepsilon\right]\!\right]\quad\text{at}\quad z=0.\end{aligned} \tag{A.15}$$

The third order problem for $O(\delta^3)$ is given by

$$L\left[\Phi_3^{(1),(2)},\Psi_3^{(1),(2)}\right]=-\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}+2\frac{\partial^2}{\partial x_0\partial x_1}+2\frac{\partial^2}{\partial y_0\partial y_1}+2\frac{\partial^2}{\partial x_0\partial x_2}+2\frac{\partial^2}{\partial y_0\partial y_2}\right]\left(\Phi_1^{(1),(2)},\Psi_1^{(1),(2)}\right) \tag{A.16}$$

$$\begin{aligned}M\left[\eta_3,\Phi_3^{(1),(2)}\right]=&-\frac{\partial\eta_2}{\partial t_1}-\frac{\partial\eta_1}{\partial t_2}+\eta_2\frac{\partial^2\Phi_1^{(1),(2)}}{\partial z^2}+\frac{\eta_1^2}{2}\frac{\partial^3\Phi_1^{(1),(2)}}{\partial z^3}-\frac{\partial\eta_1}{\partial x_0}\left\{\frac{\partial\Phi_2^{(1),(2)}}{\partial x_0}+\frac{\partial\Phi_1^{(1),(2)}}{\partial x_1}\right\}\\&-\frac{\partial\eta_1}{\partial y_0}\left\{\frac{\partial\Phi_2^{(1),(2)}}{\partial y_0}+\frac{\partial\Phi_1^{(1),(2)}}{\partial y_1}\right\}+\eta_1\frac{\partial^2\Phi_2^{(1),(2)}}{\partial z^2}-\eta_1\left[\frac{\partial\eta_1}{\partial x_0}\left(\frac{\partial^2\Phi_1^{(1),(2)}}{\partial x_0\partial z}\right)+\frac{\partial\eta_1}{\partial y_0}\left(\frac{\partial^2\Phi_1^{(1),(2)}}{\partial y_0\partial z}\right)\right]\\&-\frac{\partial\eta_2}{\partial x_0}\left(\frac{\partial\Phi_1^{(1),(2)}}{\partial x_0}\right)-\frac{\partial\eta_2}{\partial y_0}\left(\frac{\partial\Phi_1^{(1),(2)}}{\partial y_0}\right)-\frac{\partial\eta_1}{\partial x_1}\left(\frac{\partial\Phi_1^{(1),(2)}}{\partial x_0}\right)-\frac{\partial\eta_1}{\partial y_1}\left(\frac{\partial\Phi_1^{(1),(2)}}{\partial y_0}\right),\ \text{at}\quad z=0\end{aligned} \tag{A.17}$$

$$\begin{aligned}O_{1,2}[\Psi_3]=&-\left[\!\left[\frac{\partial\Psi_2}{\partial(x_1,y_1)}+\frac{\partial\Psi_1}{\partial(x_2,y_2)}\right]\!\right]-\eta_1\frac{\partial\eta_1}{\partial(x_0,y_0)}\left[\!\left[\frac{\partial^2\Psi_1}{\partial z^2}\right]\!\right]-\left(\frac{\partial\eta_2}{\partial(x_0,y_0)}+\frac{\partial\eta_1}{\partial(x_1,y_1)}\right)\\&\times\left[\!\left[\frac{\partial\Psi_1}{\partial z}\right]\!\right]-\frac{\partial\eta_1}{\partial(x_0,y_0)}\left[\!\left[\frac{\partial\Psi_2}{\partial z}\right]\!\right]-\eta_2\left[\!\left[\frac{\partial^2\Psi_1}{\partial(x_0,y_0)\partial z}\right]\!\right]-\eta_1\left[\!\left[\frac{\partial^2\Psi_2}{\partial(x_0,y_0)\partial z}\right]\!\right]\\&-\eta_1\left[\!\left[\frac{\partial^2\Psi_1}{\partial(x_1,y_1)\partial z}\right]\!\right]-\frac{\eta_1^2}{2}\left[\!\left[\frac{\partial^3\Psi_1}{\partial(x_0,y_0)\partial z^2}\right]\!\right],\ \text{at}\quad z=0\end{aligned} \tag{A.18}$$

$$\begin{aligned}G[\eta_3,\Psi_3]=&\,\eta_1\frac{\partial\eta_1}{\partial x_0}\left[\!\left[\varepsilon\frac{\partial^2\Psi_1}{\partial x_0\partial z}\right]\!\right]+\left(\frac{\partial\eta_2}{\partial x_0}+\frac{\partial\eta_1}{\partial x_1}\right)\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial x_0}\right]\!\right]+\frac{\partial\eta_1}{\partial x_0}\left[\!\left[\varepsilon\left(\frac{\partial\Psi_2}{\partial x_0}+\frac{\partial\Psi_1}{\partial x_1}\right)\right]\!\right]+\eta_1\frac{\partial\eta_1}{\partial y_0}\left[\!\left[\varepsilon\frac{\partial^2\Psi_1}{\partial y_0\partial z}\right]\!\right]\\&+\left(\frac{\partial\eta_2}{\partial y_0}+\frac{\partial\eta_1}{\partial y_1}\right)\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial y_0}\right]\!\right]+\frac{\partial\eta_1}{\partial y_0}\left[\!\left[\varepsilon\left(\frac{\partial\Psi_2}{\partial y_0}+\frac{\partial\Psi_1}{\partial y_1}\right)\right]\!\right]-\eta_2\left[\!\left[\varepsilon\frac{\partial^2\Psi_1}{\partial z^2}\right]\!\right]-\frac{\eta_1^2}{2}\left[\!\left[\frac{\partial^3\Psi_1}{\partial z^3}\right]\!\right]\\&-\eta_1\left[\!\left[\varepsilon\frac{\partial^2\Psi_2}{\partial z^2}\right]\!\right]-\left(\frac{\partial\eta_2}{\partial x_1}+\frac{\partial\eta_1}{\partial x_2}\right)E_0\left[\!\left[\varepsilon\right]\!\right],\ \text{at}\quad z=0\end{aligned} \tag{A.19}$$

$$N[\eta_3,\Phi_3,\Psi_3] = -\left\{\frac{\partial\Phi_2^{(1)}}{\partial t_1}-\rho\frac{\partial\Phi_2^{(2)}}{\partial t_1}\right\}-\left\{\frac{\partial\Phi_1^{(1)}}{\partial t_2}-\rho\frac{\partial\Phi_1^{(2)}}{\partial t_2}\right\}-\eta_2\left[\frac{\partial^2\Phi_1^{(1)}}{\partial t_0\partial z}-\rho\frac{\partial^2\Phi_1^{(2)}}{\partial t_0\partial z}\right]-\eta_1\left[\frac{\partial^2\Phi_2^{(1)}}{\partial t_0\partial z}-\rho\frac{\partial^2\Phi_2^{(2)}}{\partial t_0\partial y}\right]$$
$$-\eta_1\left[\frac{\partial^2\Phi_1^{(1)}}{\partial t_1\partial z}-\rho\frac{\partial^2\Phi_1^{(2)}}{\partial t_1\partial y}\right]-\frac{\eta_1^2}{2}\left[\frac{\partial^3\Phi_1^{(1)}}{\partial t_0\partial z^2}-\rho\frac{\partial^3\Phi_1^{(2)}}{\partial t_0\partial z^2}\right]-\eta_1\left\{\frac{\partial\Phi_1^{(1)}}{\partial x_0}\frac{\partial^2\Phi_1^{(1)}}{\partial x_0\partial z}-\rho\frac{\partial\Phi_1^{(2)}}{\partial x_0}\frac{\partial^2\Phi_1^{(2)}}{\partial x_0\partial z}\right\}$$
$$-\eta_1\left\{\frac{\partial\Phi_1^{(1)}}{\partial y_0}\frac{\partial^2\Phi_1^{(1)}}{\partial y_0\partial z}-\rho\frac{\partial\Phi_1^{(2)}}{\partial y_0}\frac{\partial^2\Phi_1^{(2)}}{\partial y_0\partial z}\right\}-\eta_1\left\{\frac{\partial\Phi_1^{(1)}}{\partial z}\frac{\partial^2\Phi_1^{(1)}}{\partial z^2}-\rho\frac{\partial\Phi_1^{(2)}}{\partial z}\frac{\partial^2\Phi_1^{(2)}}{\partial z^2}\right\}-\left\{\frac{\partial\Phi_1^{(1)}}{\partial x_0}\frac{\partial\Phi_2^{(1)}}{\partial x_0}-\rho\frac{\partial\Phi_1^{(2)}}{\partial x_0}\frac{\partial\Phi_2^{(2)}}{\partial x_0}\right\}$$
$$-\left\{\frac{\partial\Phi_1^{(1)}}{\partial y_0}\frac{\partial\Phi_2^{(1)}}{\partial y_0}-\rho\frac{\partial\Phi_1^{(2)}}{\partial y_0}\frac{\partial\Phi_2^{(2)}}{\partial y_0}\right\}-\left\{\frac{\partial\Phi_1^{(1)}}{\partial x_0}\frac{\partial\Phi_1^{(1)}}{\partial x_1}-\rho\frac{\partial\Phi_1^{(2)}}{\partial x_0}\frac{\partial\Phi_1^{(2)}}{\partial x_1}\right\}-\left\{\frac{\partial\Phi_1^{(1)}}{\partial y_0}\frac{\partial\Phi_1^{(1)}}{\partial y_1}-\rho\frac{\partial\Phi_1^{(2)}}{\partial y_0}\frac{\partial\Phi_1^{(2)}}{\partial y_1}\right\}$$
$$-\left\{\frac{\partial\Phi_1^{(1)}}{\partial z}\frac{\partial\Phi_2^{(1)}}{\partial z}-\rho\frac{\partial\Phi_1^{(2)}}{\partial z}\frac{\partial\Phi_2^{(2)}}{\partial z}\right\}+\frac{\partial^2\eta_1}{\partial x_1^2}+\frac{\partial^2\eta_1}{\partial y_1^2}+2\left[\frac{\partial}{\partial x_0}\left\{\frac{\partial\eta_2}{\partial x_1}+\frac{\partial\eta_1}{\partial x_2}\right\}+\frac{\partial}{\partial y_0}\left\{\frac{\partial\eta_2}{\partial y_1}+\frac{\partial\eta_1}{\partial y_2}\right\}\right]-2\frac{\partial\eta_1}{\partial x_0}\frac{\partial\eta_1}{\partial y_0}\frac{\partial^2\eta_1}{\partial x_0\partial y_0}$$
$$-\frac{1}{2}\frac{\partial^2\eta_1}{\partial x_0^2}\left(\frac{\partial\eta_1}{\partial y_0}\right)^2-\frac{1}{2}\frac{\partial^2\eta_1}{\partial y_0^2}\left(\frac{\partial\eta_1}{\partial x_0}\right)^2-\frac{3}{2}\left[\frac{\partial^2\eta_1}{\partial x_0^2}\left(\frac{\partial\eta_1}{\partial x_0}\right)^2+\frac{\partial^2\eta_1}{\partial y_0^2}\left(\frac{\partial\eta_1}{\partial y_0}\right)^2\right]-\eta_1\left[\!\left[\varepsilon\left\{\frac{\partial\Psi_1}{\partial x_0}\left(\frac{\partial^2\Psi_1}{\partial x_0\partial z}\right)+\frac{\partial\Psi_1}{\partial y_0}\left(\frac{\partial^2\Psi_1}{\partial y_0\partial z}\right)\right\}\right]\!\right]$$
$$-\left[\!\left[\varepsilon\left(\frac{\partial\Psi_1}{\partial x_0}\right)\left\{\frac{\partial\Psi_2}{\partial x_0}+\frac{\partial\Psi_1}{\partial x_1}\right\}\right]\!\right]-\left[\!\left[\varepsilon\left(\frac{\partial\Psi_1}{\partial y_0}\right)\left\{\frac{\partial\Psi_2}{\partial y_0}+\frac{\partial\Psi_1}{\partial y_1}\right\}\right]\!\right]+\eta_1\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial z}\left(\frac{\partial^2\Psi_1}{\partial z^2}\right)\right]\!\right]+\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial z}\left(\frac{\partial\Psi_2}{\partial z}\right)\right]\!\right]+2E_0\frac{\partial\eta_1}{\partial x_0}\left[\!\left[\varepsilon\frac{\partial\Psi_2}{\partial z}\right]\!\right]$$
$$+\eta_1E_0\left[\!\left[\varepsilon\frac{\partial}{\partial z}\left(\frac{\partial\Psi_2}{\partial x_0}+\frac{\partial\Psi_1}{\partial x_1}\right)\right]\!\right]+\eta_2E_0\left[\!\left[\varepsilon\left(\frac{\partial^2\Psi_1}{\partial x_0\partial z}\right)\right]\!\right]+\frac{\eta_1^2}{2}E_0\left[\!\left[\varepsilon\left(\frac{\partial^3\Psi_1}{\partial x_0\partial z^2}\right)\right]\!\right]+E_0\left[\!\left[\varepsilon\left(\frac{\partial\Psi_2}{\partial x_1}+\frac{\partial\Psi_1}{\partial x_2}\right)\right]\!\right]$$
$$+2\eta_1E_0\frac{\partial\eta_1}{\partial x_0}\left[\!\left[\varepsilon\frac{\partial^2\Psi_1}{\partial z^2}\right]\!\right]+2E_0\left\{\frac{\partial\eta_2}{\partial x_0}+\frac{\partial\eta_1}{\partial x_1}\right\}\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial z}\right]\!\right]-2E_0\left(\frac{\partial\eta_1}{\partial x_0}\right)\left(\frac{\partial\eta_1}{\partial y_0}\right)\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial y_0}\right]\!\right]-2\frac{\partial\eta_1}{\partial x_0}\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial x_0}\left(\frac{\partial\Psi_1}{\partial z}\right)\right]\!\right]$$
$$-2\frac{\partial\eta_1}{\partial y_0}\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial y_0}\left(\frac{\partial\Psi_1}{\partial z}\right)\right]\!\right]+2\frac{\partial\eta_1}{\partial x_0}\left(\frac{\partial\eta_2}{\partial x_0}+\frac{\partial\eta_1}{\partial x_1}\right)E_0^2\,[\varepsilon]-2E_0\left(\frac{\partial\eta_1}{\partial x_0}\right)^2\left[\!\left[\varepsilon\frac{\partial\Psi_1}{\partial x_0}\right]\!\right],\quad \text{at}\quad z=0. \qquad \text{(A.20)}$$

(*See equation (A.20) above.*)

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